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On the General Solution of Laplace's Equation and the Equation of Wave Motions, and on an undulatory explanation of Gravity. By E. T. Whittaker, M.A.

1. *Laplace's Equation.*

The first part of this communication is concerned with the well-known equation which was first introduced in Laplace's memoir on the ring of *Saturn*, and which is now of such importance in mathematical physics. The general solution has not hitherto been found in a finite form: we shall show that it is given in a finite form by the following result:

The general solution of Laplace's equation

$$\frac{\partial^2 V}{\partial x^2}+\frac{\partial^2 V}{\partial y^2}+\frac{\partial^2 V}{\partial z^2}=0$$

is

$$V=\int_0^{2\pi} f(x\cos v+y\sin v+iz,\, v)\, dv,$$

where f is an arbitrary function of the two arguments $x\cos v+y \sin v+iz$ and v.

To establish this, let $V(x, y, z)$ be any solution (single-valued or many-valued) of Laplace's equation. Since the differential equation is unaffected by a change of origin, and since any function of the two arguments $(x \cos v+y \sin v+iz)$ and v is also a function of the arguments $(x' \cos v+y' \sin v+iz')$ and v, where (x', y', z') are the coordinates of the point (x, y, z) referred to parallel axes through any other origin, it clearly will not affect the generality of the proof if any particular point be taken as origin; we shall therefore suppose that the origin is taken at some point at which some branch of the function $V(x, y, z)$ is

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regular, so that this branch of V (x, y, z) can be expanded in the neighbourhood of the origin in the form

$$V(x, y, z) = a_0 + a_1 x + b_1 y + c_1 z + a_2 x^2 + b_2 y^2 + c_2 z^2 + d_2 yz + e_2 zx + f_2 xy + a_3 x^3 + \ldots,$$

where $a_0, a_1, b_1, \ldots$ are constants, and the series is absolutely and uniformly convergent for a domain of finite extent.

Substituting this expansion in Laplace's equation, and equating to zero the coefficients of the various powers of x, y, and z, we obtain an infinite number of linear relations between the constants $a_0, a_1, b_1, \ldots$ There are $\frac{1}{2}n(n-1)$ of these relations between the $\frac{1}{2}(n+1)(n+2)$ coefficients of the terms of any degree n in the expansion of V; so that only $\{\frac{1}{2}(n+1)(n+2) - \frac{1}{2}n(n-1)\}$ or $(2n+1)$ of the coefficients of terms of degree n in the expansion of V are really independent.

Moreover, it is clear that the terms of degree n in V must satisfy Laplace's equation, quite independently of the terms of other degrees; and therefore the terms of degree n in V must be a linear combination of $(2n+1)$ linearly independent solutions of degree n.

To find these solutions, consider the expansion of the quantity $(x \cos v + y \sin v + iz)^n$ as a sum of sines and cosines of multiples of v, in the form

$$(x \cos v + y \sin v + iz)^n = g_0(x, y, z) + g_1(x, y, z) \cos v + g_2(x, y, z) \cos 2v + \ldots + g_n(x, y, z) \cos nv + h_1(x, y, z) \sin v + h_2(x, y, z) \sin 2v + \ldots + h_n(x, y, z) \sin nv.$$

The $(2n+1)$ quantities $g(x, y, z)$ and $h(x, y, z)$ are clearly linearly independent homogeneous polynomials of degree n in x, y, z; and each of them must satisfy Laplace's equation, since $(x \cos v + y \sin v + iz)^n$ does so. These are therefore the $(2n+1)$ independent solutions of Laplace's equation, of degree n.

Now since by Fourier's theorem we have the relations

$$g_m(x, y, z) = \frac{1}{\pi}\int_0^{2\pi} (x \cos v + y \sin v + iz)^n \cos mv \, dv$$

$$h_m(x, y, z) = \frac{1}{\pi}\int_0^{2\pi} (x \cos v + y \sin v + iz)^n \sin mv \, dv,$$

it follows that each of these $(2n+1)$ solutions can be expressed in the form

$$\int_0^{2\pi} f(x \cos v + y \sin v + iz, v) \, dv,$$

and therefore any linear combination of these $(2n+1)$ solutions can be expressed in this form. That is, the terms of degree n in V (x, y, z) can be expressed in this form, where n is any positive integer; and therefore V (x, y, z) (which was taken to be *any* solution of Laplace's equation) is itself of this form; which establishes the theorem.

2. *The Partial Differential Equation of Wave-Motions.*

It can in the same way be shown that the *general solution of the general differential equation of wave-motions, namely*

$$\frac{\partial^2 V}{\partial x^2}+\frac{\partial^2 V}{\partial y^2}+\frac{\partial^2 V}{\partial z^2}=k^2\frac{\partial^2 V}{\partial t^2}$$

is

$$V=\int_0^{\pi}\int_0^{2\pi} f\,(x \sin v \cos \psi + y \sin v \sin \psi + z \cos v + kt,\ v,\ \psi)\, dv\, d\psi,$$

where f is an arbitrary function of the three arguments

x sin v cos $\psi + y$ sin v sin $\psi + z$ cos $v + kt$, v, and ψ.

From this result it follows, by the application of Fourier's double-integral theorem, that *any solution of the equation*

$$\frac{\partial^2 V}{\partial x^2}+\frac{\partial^2 V}{\partial y^2}+\frac{\partial^2 V}{\partial z^2}=k^2\frac{\partial^2 V}{\partial t^2}$$

can be analysed into simple plane waves.

3. *An Undulatory Explanation of Gravity.*

In connection with the last property it can be shown that *distributions of simple waves exist such that, although each individual wave is periodic with respect to the time, yet the total disturbance at any point does not vary with the time.* The simplest example of this is the following. Suppose a particle to be emitting spherical waves, such that the disturbance due to the waves whose wave-length lies between $\frac{2\pi}{\mu}$ and $\frac{2\pi}{\mu+d\mu}$ is

$$\frac{2d\mu}{\pi\mu}\cdot\frac{\sin \mu(Vt-r)}{r},$$

where V is the velocity of propagation of the waves. Then after the waves have reached the point r, so that $(Vt-r)$ is positive, the total disturbance there (due to the sum of all the waves) is

$$\frac{2}{\pi r}\int_0^{\infty}\frac{\sin \mu(Vt-r)}{\mu}\,d\mu$$

or

$$\frac{2}{\pi r}\int_0^{\infty}\frac{\sin y}{y}\,dy, \quad \text{where } \mu(Vt-r)=y,$$

or

$$\frac{1}{r},$$

i.e. the disturbance is independent of the time and is everywhere proportional to the gravitational potential of the particle. It is easily seen that *for all such distributions the total disturbance can always be represented by a solution of Laplace's equation.*

Since Laplace's equation is the equation satisfied by the components of any field of force due to gravitating bodies, this result suggests an explanation of the propagation of gravitation. For we thus see that it is possible to analyse the field of force due to a gravitating body, by a "spectrum analysis" as it were, into an infinite number of constituent fields; and although the whole field of force does not vary with the time, yet each of the constituent fields is of an undulatory character, *i.e.* the disturbance at any point reproduces itself after the lapse of any time at another point whose distance is proportional to the elapsed time. But this result assimilates the propagation of gravitation to that of light; for the undulatory phenomena just described, in which the varying vector is a gravitational force perpendicular to the wave-front, may be compared with those made familiar by the electromagnetic theory of light, in which the varying vectors consist of electric and magnetic forces parallel to the wave-front. The waves are in other respects of the same type in the two cases, and it seems probable that an identical property of the medium ensures their transmission through space. This undulatory theory of gravity would require that gravity should be propagated with a finite velocity, which, however, need not be the same as that of light, and may be enormously greater.

Of course this investigation does not explain the *cause* of gravity; all that is done is to show that the propagation across space of forces which vary as the inverse square of the distance does not require for its explanation any other property of the medium than one which has long been known and accepted.

On the Mean Distance of a Planet, as a Function of Three Heliocentric Distances and the Observed Times. By Shin Hirayama.

§ 1. Let r_1, r_2, r_3 be three heliocentric distances of a planet;
t_1, t_2, t_3 the corresponding observed times;
a the mean distance.

The radius vector of a planet must satisfy the differential equation

$$\frac{d^2s}{dt^2} - \frac{2}{\sqrt{s}} + \frac{2}{a} = 0,$$

putting $r^2 = s$.

If we let

$$\theta_1 = k(t_3 - t_2),\ \theta_2 = k(t_3 - t_1),\ \theta_3 = k(t_2 - t_1),\ \log k = 8{\cdot}235581 - 10,$$

and express the value r^2 in terms of t by expansion into series, we have

$$s = r^2 = a_0 + a_1 t + a_2 t^2 + a_3 t^3 + a_4 t^4,$$

where a_0 a_1, a_2, a_3, a_4 are five constants independent of time.